Title: Existence of complete Kähler-Einstein metric with negative scalar curvature

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By Liouville's theorem in complex analysis, the complex plane $\mathbb{C}$ contains no nonconstant bounded holomorphic functions. But the unit disk $\mathbb{D}$ in $\mathbb{C}$ has plenty of bounded holomorphic functions.

From geometric viewpoint, $\mathbb{C}$ has no metrics of negative curvature. The unit disk $\mathbb{D}$ has Poincaré metric

$$\omega_{\mathcal{P}} = \frac{dz \otimes d\bar{z}}{(1 - |z|^2)^2}$$

which has curvature -1.

To generalize to a complex manifold M, the holomorphic function is naturally replaced by the holomorphic section of canonical bundle K_M. Such a holomorphic section is locally given by

$$f(z)dz^1 \wedge \cdots \wedge dz^n,$$

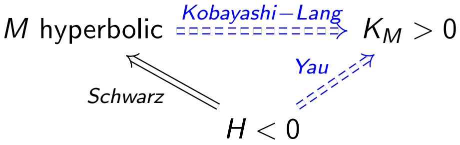
where $n = \dim_{\mathbb{C}} M$. A natural question is, when does K_M have plenty of holomorphic sections?

A classical result of Kodaira tells us that, if M is compact Kähler with $K_M > 0$, then K_M is ample; in particular, for all sufficiently large m, the bundle mK_M have lots of holomorphic sections. Now the question is, when does a complex manifold M have $K_M > 0$?

A well-known theorem of S. S. Chern says that if M admits a metric with negative Ricci curvature, then $K_M > 0$.

Since the proof of Calabi conjecture in the case of negative scalar curvature due to Aubin and myself independently, we know that a $K_M > 0$ if and only if M admits a Kähler-Einstein metric with negative scalar curvature.

In the late 1970s I conjecture that if a compact complex manifold M has negative holomorphic sectional curvature H, then $K_M > 0$. On the other hand, the conjectures of Kobayashi and Lang assert that if M is Kobayashi hyperbolic then $K_M > 0$.



These conjectures are related via the Schwarz Lemma. Negative holomorphic curvature on a compact manifold M implies M is Kobayashi hyperbolic. Its converse does not hold in general, in view of the example of Demailly (1995).

Both Kobayashi-Lang conjecture and my conjecture would hold for submanifolds provide they hold for the ambient manifolds, because of the decreasing property of holomorphic sectional curvature and the Kobayashi hyperbolicity.

Several authors have made contributions to these conjectures. In complex dimension two, these are answered affirmatively independently by Bun Wong and Campana, by means of the classification theory of compact complex surfaces. A short direct proof is later provided by the join paper of Godon Heier, Steven Lu, and Bun Wong (2010) using only standard algebraic geometry (Nakai-Moishezon-Kleiman criterion, Riemann-Roch, and Hodge Index Theorem) and a generalized Gauss-Bonnet theorem due to Bishop-Goldberg.

In higher dimensions, it is natural to first consider the projective algebraic manifolds, where some algebraic-geometric tools and partial classifications are available. Peternell (1991) proves the Kobayash-Lang conjecture for projective three-fold except for the Calabi-Yau threefold which contain no rational curves.

As a testing case, Pit-Mann Wong, Damin Wu and myself several years ago prove my conjecture for all projective manifolds with Picard number equal to one. The holomorphic sectional curvature in Wong-Wu-Yau is only assumed to be quasi-negative (i.e., nonpositive everywhere and negative at one point).

The projective threefold case of my conjecture has been completely settled by a series of paper of Gordon Heier, Steven Lu, and Bun Wong (2010, 2014), which make an interesting connection to the abundance conjecture in the algebraic geometry. They indeed prove my conjecture by assuming the validity of the abundance conjecture, which is known to hold for dimension less than four.

More precisely, Heier-Lu-Wong prove that if a projective manifold with negative holomorphic sectional curvature then the canonical bundle is nef, and the nef dimension is equal the dimension of the manifold. (The nef dimension of a nef class on X is equal to $\dim Y$, where there exists a dominant rational map $X \dashrightarrow Y$ satisfying certain numerical properties and Y is unique up to a birational map.) A version of the abundance conjecture asserts that for a projective manifold with nef canonical bundle, the Kodaira dimension is equal to the nef dimension, that is, the manifold is of general type.

Recently, Damin Wu and myself are able to remove the need for the abundance conjecture. Wu-Yau (2015) provides two slightly different proofs for my conjecture for the projective manifolds in all dimensions. That is, if a projective manifold M admits a Kähler metric with negative holomorphic sectional curvature, then K_M is ample.

In view of the decreasing property of holomorphic sectional curvature, we know any smooth subvariety in M also has positive canonical bundle. In particular, In particular, every nonsingular subvariety of a smooth compact quotient of the unit ball in $\mathbb{C}^n$ has ample canonical bundle.

The first proof in Wu-Yau (2015) reduces to show the integral inequality

$$\int_M c_1(K_M)^n > 0.$$

In fact, the hyperbolicity implies M contains no rational curve; by Mori's theory, K_M is nef. The nefness together with the integral inequality implies K_M is big, which is due to the result of Demailly, Siu, Trapani, and other people, as an application of Demailly's holomorphic Morse inequality.

A important step in Wu-Yau (2015) is to introduce a Monge-Ampere type equation to construct a family of Kähler metrics whose Ricci curvature has a lower bound. This allows one to apply the refined Schwarz Lemma to show the desired integral inequality.

The refined Schwarz Lemma is initiated by Ahlfors (1938), developed by Chern (1960s), myself in 1978, Royden (1980), and many other people. The version used in Wu-Yau (2015) is a strengthen result of our previous work joint with Fangyang Zheng, and with Pit-Mann Wong.

The second proof in Wu-Yau (2015) is to directly show that the family of metrics converges to a Kähler-Einstein metric, which implies $K_M > 0$.

By using both the Monge-Ampère equation and the refined Schwarz Lemma in Wu-Yau (2015), Tosatti-Yang (2015) show that if a Kähler manifold has nonpositive holomorphic sectional curvature, the canonical bundle is nef. This combining the second proof in Wu-Yau (2015) enables them to extend our result to the Kähler manifolds.

Wu and I have realized that there is a direct proof of my conjecture in the Kähler case, which uses purely geometric analysis, bypassing the notion of nefness. The proof will be given shortly.

It is natural to extend these results to the case the holomorphic sectional curvature H is quasi-negative, as in Wong-Wu-Yau (2010). This extension is established by Deverio-Trapani (2016) and Wu-Yau (2016), using the Monge-Ampère type equation and the refined Schwarz lemma. In this situation, the key is the compactness argument. Deverio-Trapani uses the pluripotential theory, while Wu-Yau uses an elementary lemma inspired by the work of S. Y. Cheng and myself in the mid 1970s.

The Monge-Ampère type equation and the refined Schwarz lemma can in fact be used to study the ampleness under the Carathéodory hyperbolicity. We can prove the following result: If a compact Kähler manifold M is Carathéodory hyperbolic, then $K_M > 0$. Here a complex manifold M is called Carathéodory hyperbolic if there exists a holomorphic covering $\widetilde{M}$ of M such that the Carathéodory pseudo-metric on $\widetilde{M}$ is a metric.

The Carathéodory pseudo-metric on $\widetilde{M}$ can be defined as below. Given any complex holomorphic vector X at a point, look at all holomorphic maps F mapping $\widetilde{M}$ to the unit disk so that F maps X to a vector at the origin. We maximize the Poincaré length of $F(X)$, which is the Carathéodory pseudo-metric of X.

The point is now the metric that we had can be lifted to its universal cover. Then apply Schwarz lemma to show that it is dominated from below by the Carathéodory metric. We shall discuss a more general result in the complete Kähler setting.

We now summarize the recent results and give proofs of them.

Theorem (Wu-Yau (2015), Tosatti-Yang (2015), Deverio-Trapani (2016), Wu–Yau (2016))

Let (M, ω) be a compact Kähler manifold, and $H(\omega)$ be the holomorphic sectional curvature of ω.

(i) *If $H(\omega) < 0$ everywhere on M, then $K_M > 0$.*

(ii) *If $H(\omega) \leq 0$ everywhere on M, then K_M is nef.*

(iii) *If $H(\omega)$ is quasi-negative, i.e., $H(\omega) \leq 0$ everywhere and $H(\omega) < 0$ at one point of M, then $K_M > 0$.*

Analytic proof for (i) $H < 0 \Longrightarrow K_M > 0$.

Let (M, ω) be a Kähler manifold. Inspired by the nefness of K_M, we consider

$$
\begin{aligned}
(t\omega + dd^c \log \omega^n + dd^c u)^n &= e^u \omega^n, \quad t \geq 0, \\
\omega_t \equiv t\omega + dd^c \log \omega^n + dd^c u &> 0.
\end{aligned}
$$

Here ω is the background Kähler metric with negative holomorphic sectional curvature, and $dd^c \log \omega^n$ is the Chern form representing the first Chern class of K_M.

We would like to solve u for $t = 0$, by the continuity method. First, we claim that for a sufficiently large t_1, the equation has a smooth solution.

To see this, we can pick a large t_1 such that $t_1\omega + dd^c \log \omega^n$ is positive definite on the compact manifold. Then $t\omega + dd^c \log \omega^n$ defines a Kähler metric since it is d-closed. Note that the equation can be rewritten as

$$(t_1\omega + dd^c \log \omega^n + dd^c u)^n = e^{u+f}(t_1\omega + dd^c \log \omega^n)^n,$$

where f is a smooth function given by

$$f = \log \frac{\omega^n}{(t_1\omega + dd^c \log \omega^n)^n}.$$

This equation has a smooth solution u, by my early work on the Calabi conjecture. This proves the claim.

Let

$$I = \{t \in [0, t_1]; \omega_t^n = e^u \omega^n, \omega_t > 0\}.$$

Then I is not empty, since $t_1 \in I$. To see I is open in $[0, t_1]$, let $t_0 \in I$ with solution u_{t_0}. Define

$$\mathcal{M}(t, v) = \log \frac{(t\omega + dd^c \log \omega^n + dd^c v)^n}{\omega^n} - v$$

for all (t, v) in a near (t_0, u_{t_0}). Then $\mathcal{M}(t_0, u_{t_0}) = 0$ and the linearization of $\mathcal{M}$ at (t_0, u_{t_0}) is

$$\Delta_{\omega_{t_0}} - 1,$$

which is invertible between the Hölder spaces. Thus, applying the implicit function theorem yields the openness of I in $[0, t_1]$.

The closedness of I requires the Schwarz Lemma. We use the following version of Schwarz Lemma, based on Wu-Yau-Zheng (2009) and Wong-Wu-Yau (2010).

lemma (Yau (1978), Royden (1980), Wu-Yau (2015))

Let M^n be a complex manifold with two Kähler metrics ω_1 and ω_2. If $H(\omega_1) \leq -\kappa$ and $\mathrm{Ric}(\omega_2) \geq \lambda\omega_2 + \mu\omega_1$, then

$$\Delta_{\omega_2} \log(\mathrm{tr}_{\omega_2}\omega_1) \geq \Big(\frac{(n+1)\kappa}{2n} + \frac{\mu}{n}\Big)\mathrm{tr}_{\omega_2}\omega_1 + \lambda.$$

Here λ, κ, μ are continuous functions on M and $\kappa \geq 0$, $\mu \geq 0$ on M.

Closedness

The key feature of the Monge-Ampère equation is that $\omega_t^n = e^u \omega^n$ implies

$$\mathrm{Ric}(\omega_t) = -\omega_t + t\omega, \quad \sup u \le C.$$

Since $H(\omega) < 0$ and M is compact, $H(\omega) \le -\kappa$ for some constant $\kappa > 0$.
Applying the Schwarz lemma $\omega_1 = \omega$ and $\omega_2 = \omega_t$ yields

$$\Delta_{\omega_2} \log(\mathrm{tr}_{\omega_t}\omega) \ge \Big(\frac{(n+1)\kappa}{2n} + \frac{t}{n}\Big)\mathrm{tr}_{\omega_t}\omega - 1.$$

By the maximum principle,

$$\mathrm{tr}_{\omega_t}\omega \le \frac{2n}{(n+1)\kappa}.$$

The estimates on $\text{tr}_{\omega_t}\omega$ and $\sup u$ are sufficient for the closedness of I, by the argument in my early work. One way to see this is as below. We can normalize at one point such that the components of ω satisfies $g_{i\bar{j}} = \delta_{ij}$ and the components of ω_t satisfies $g'_{i\bar{j}} = \lambda_i \delta_{ij}$. The bound on $\sup u$ yields

$$\lambda_1 \cdots \lambda_n \leq C$$

while the bound on $\text{tr}_{\omega_t}\omega$ is

$$\text{tr}_{\omega_t}\omega = \frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n} \leq C.$$

By the elementary inequality

$$\sum_{i=1}^{n} \lambda_i \leq (\mathrm{tr}_{\omega_t}\omega)^{n-1} \prod_{i=1}^{n} \lambda_i \leq C.$$

Hence,

$$C^{-1} \leq \lambda_i \leq C, \quad i = 1, \dots, n.$$

This also gives a lower bound on u, as

$$e^u = \lambda_1 \cdots \lambda_n \geq C^{-n}.$$

Thus, we obtain

$$C^{-1}\omega \leq \omega_t \leq C\omega,$$

as well as the estimates of u up to the second order.

The third order estimate of u can be derived in a similar way as in my early work: Let $Y \equiv g'_{i\bar{j};k} g'_{\bar{r}a;\bar{b}} g'^{i\bar{r}} g'^{a\bar{j}} g'^{k\bar{b}}$ to get

$$\Delta_{\omega_t}(Y + C\Delta_\omega u) \geq C_1(Y + C\Delta_\omega u) - C_2.$$

Hence, by the maximum principle,

$$Y \leq C.$$

Now letting $t \to 0$ we get a smooth solution u_* satisfying

$$(dd^c \log \omega^n + dd^c u_*)^n = e^{u_*}\omega^n,$$
$$dd^c \log \omega^n + dd^c u_* > 0,$$

which gives the desired Kähler-Einstein metric with negative scalar curvature. This in particular implies the canonical bundle is positive.

Analytic proof of (ii) $H \leq 0 \Rightarrow K_M$ nef.

It is sufficient to solve the Monge-Ampère type equation

$$\omega_t^n = (t\omega + dd^c \log \omega^n + dd^c u)^n = e^u \omega^n, \quad \omega_t > 0,$$

for every small $t > 0$. Again use the continuity method: The nonemptyness, openness, and C^0 estimate are the same as (i). Only difference is that the upper bound κ of $H(\omega)$ can be zero. Now the Schwarz Lemma reads

$$\Delta_{\omega_t} \log(\mathrm{tr}_{\omega_t}\omega) \geq \frac{t}{n}\mathrm{tr}_{\omega_t}\omega - 1.$$

It follows that

$$\mathrm{tr}_{\omega_t}\omega \leq \frac{n}{t} \leq \frac{n}{t_2} \quad \text{for all } t_2 \leq t \leq t_1.$$

Here $t_2 > 0$ is arbitrary. The C^2 estimate becomes

$$t_2 C^{-1}\omega \leq \omega_t \leq C t_2^{1-n}\omega.$$

The higher order estimates of u depends on t_2. Since $t_2 > 0$ is arbitrary, we obtain a smooth solution u of $\omega_t^n = e^u \omega^n$. In particular, $e^u \omega^n$ gives rise to a smooth metric on K_M so that its curvature form

$$dd^c \log(e^u \omega^n) = \omega_t - t\omega > -t\omega.$$

This implies K_M is nef.

Sketch proof of (iii) H quasi-negative $\Rightarrow K_M > 0$.

Since $H(\omega) \leq 0$ implies that K_M is nef and M contains no rational curve, it is sufficient to show K_M is big, i.e.,

$$\int_X c_1(K_M)^n > 0.$$

Note that

$$\int_X \omega_t^n = \int_X c_1(K_M)^n + tn \int_X c_1(K_M)^{n-1} \wedge \omega + O(t^2), \quad t \to 0.$$

It suffices to find a sequence t_j such that

$$\lim_{j \to +\infty} \int_X \omega_{t_j}^n > 0.$$

Need to bound $\max u_{t_j}$ away from $-\infty$.

A compactness Lemma

The following lemma is inspired by my work with S. Y. Cheng in 1975.

lemma

Let (M,ω) be an n-dimensional compact Kähler manifold, and let v be a C^2 function satisfying $v \leq -1$ on M and

$$\Delta_\omega v \geq -C_0$$

for some constant $C_0 > 0$ on M. Then,

$$\int_M |\log(-v)|^2\omega^n + \int_M |\nabla \log(-v)|^2\omega^n \leq C\left[1 + \min_M(-v)\right]$$

where $C > 0$ is a constant depending only on n, ω, and C_0.

Note

$$\mathrm{tr}_{\omega}\omega_t > 0 \Longrightarrow \Delta_{\omega} u \geq -nt + s \geq -C_0.$$

Apply the compactness lemma to $v_t = u_t - \max u_t - 1$ to obtain a sequence $\log(-v_j)$ converges in $L^q(M)$ to w. Thus,

$$v_j \longrightarrow -e^{w} \quad \text{almost everywhere on } M.$$

Applying Schwarz lemma and elementary inequality to obtain

$$\Delta_{\omega_t} \log \mathrm{tr}_{\omega_t}\omega \geq \frac{(n+1)\kappa}{2} e^{-\max u/n} - 1.$$

Integrating against ω_t^n yields

$$\exp(-\max u_j/n) \leq \frac{2\int_M e^{v_j}\omega^n}{(n+1)\int_M \kappa e^{v_j}\omega^n} \leq C,$$

since $\kappa > 0$ in an open subset of M.

By passing to a subsequence $u_l \to -e^w + c$ almost everywhere in M. Hence,

$$\lim_{l\to+\infty} \int_M \omega_{t_l}^n = \lim_{l\to+\infty} \int_M e^{u_l}\omega^n > 0.$$

This completes the proof of the result that $H(\omega)$ being quasi-negative implies K_M is ample.

The compactness lemma can be proved as below. We compute

$$\Delta_\omega \log(-v) = \frac{-\Delta v}{-v} - |\nabla \log(-v)|^2.$$

Since $\Delta_\omega v \geq -C_0$ and $\min_M(-v) \geq 1$, integrating over M yields

$$\int_M |\nabla \log(-v)|^2 \leq C_0 \int_M \omega^n.$$

On the other hand, applying the weak Harnack inequality to $\Delta_\omega(-v) \leq C_0$ yields that, for any $1 \leq q < n/(n-1)$,

$$\Big(\int_M (-v)^q \omega^n \Big)^{1/q} \leq C \Big[1 + \min_M(-v) \Big].$$

In particular, put $q = 1$ and note $(-v) = e^{\log(-v)} \geq [\log(-v)]^2/2$. This implies the L^2 norm of $\log(-v)$. Combining these two inequalities yields the desired estimate.

Complete Kähler-Einstein metric

We seek generalize the positivity to complete noncompact Kähler manifolds. Note that on a compact Kähler manifold M, that K_M is positive is equivalent to the existence of Kähler-Einstein metric on M with negative scalar curvature. Thus, our question is, under what condition on the holomorphic sectional curvature would the complete Kähler manifold admits a complete Kähler-Einstein metric with negative scalar curvature.

The first result we proved in this direction is the following.

Theorem 1 (Wu-Yau (2017))

Let (M, ω) be a complete Kähler manifold whose holomorphic sectional curvature $H(\omega)$ satisfies $-\kappa_2 \leq H(\omega) \leq -\kappa_1$ for two constants $\kappa_1, \kappa_2 > 0$. Then, M admits a unique complete Kähler-Einstein metric ω_{KE} with Ricci curvature equal to -1. Furthermore, ω_{KE} is uniformly equivalent to ω, and the curvature tensor of ω_{KE} and all its covariant derivatives are bounded.

We consider again the Monge-Ampère equation

$$\begin{cases} (t\omega + dd^c \log \omega^n + dd^c u)^n = e^u \omega^n, \\ \omega_t \equiv t\omega + dd^c \log \omega^n + dd^c u > 0 \end{cases} \qquad (\mathrm{MA})_t$$

with the continuity method, as in the compact case. The key difference is below: Notice that the openness, nonemptyness, and bootstrap argument use the Schauder type estimate. The standard Schauder estimate requires the injectivity radius of the complete manifold to be positive, for which (M, ω) need not have. An example is the Poincaré punctured disk.

To overcome this difficulty, we need to develop the notion of (quasi-) bounded geometry initiated by S. Y. Cheng and myself more than thirty five years ago. The idea goes as follows: If the curvature tensor of the Riemannian manifold (M, ω) is bounded, then there is a constant $R > 0$, depending only on the curvature bounds, such that for any point $x \in M$, the exponential map $\exp_x$ is immersion on the ball $B(R)$ of radius R in the tangent space. Then, the pullback metric on $B(R)$ under $\exp_x$ has a nice property that its Laplacian is uniformly elliptic on the ball $B(R)$. If, in addition, the curvature tensor of ω and all its derivatives are bounded on M, we can apply the Schauder estimates to the Laplacian of $\exp_x^* \omega$ on $B(R)$.

Thus, instead of usual coordinate charts, we shall work on the quasi-coordinate charts $\{(B(R), \exp_x)\}$, which is sufficient for solving partial differential equations on manifolds. However, for our further applications to complex geometry, it is desired to have holomorphic coordinate charts $\{(B(R_1), \psi_x)\}$ for which the radius R_1 is uniformly bounded away from zero.

To produce holomorphic coordinates, one needs to solve a $\bar{\partial}$-equation. The starting point is the following inequality, established by Siu and myself forty years ago,

$$|\bar{\partial}(x^j + \sqrt{-1}\, x^{n+j})| \leq Cr^2 \quad \text{on } B(R),$$

where $x = (x^1, \ldots, x^{2n})$ is a geodesic normal coordinate system and $r = |x|$ is the Euclidean distance.

By the L^2-estimate of $\bar{\partial}$, we obtain a system of holomorphic functions which form an independent set in a small neighborhood of the origin. Nevertheless, in applications we need to have an effective version, that is, a system of holomorphic coordinates is defined in a small ball $B(R_1)$, where the radius R_1 depends only on the curvature bounds. This requires some effective estimates, which have been established in my recent work joint with Damin Wu.

Once the quasi-coordinate charts $\{(B(R_1), \psi_x)\}$ are obtained, one can define the Hölder spaces on M, as in Cheng-Yau (1980), by pulling back the functions via ψ_x to $B(r)$ and taking supremum of all Hölder norms over $B(r)$. We can now adapt the Schauder theory to these Hölder spaces. This together with my generalized maximum principle enable us to prove the nonemptyness, openness, and the bootstrap argument.

Notice that in the process of constructing the quasi-coordinate chats, we assume that the background metric ω has bounds for all the covariant derivatives of its curvature tensor. To remove the derivative bound, we invoke another key ingredient, which is a result of Wan-Xiong Shi, obtained twenty years ago.

W. X. Shi (1997) proves that if a complete Kähler manifold (M, ω) has bounded sectional curvature, then M admits another complete Kähler metric ω_1 which is uniformly equivalent to ω, and the curvature tensor of ω_1 has bounded covariant derivatives of arbitrary order. Shi's argument is to use the Ricci flow and derive its short time existence. His argument can be extended to show that if the original metric ω has negatively pinched holomorphic sectional curvature, so is the new metric ω_1.

By replacing ω with ω_1, we can assume the curvature tensor of ω has bounded covariant derivatives of arbitrary order, in addition to $H(\omega)$ negatively pinched. Then, ω has quasi-bounded geometry. This settles the nonemptyness and openness of $\omega_t^n = e^u \omega^n$. By the refined Schwarz lemma we have $\mathrm{tr}_{\omega_t}\omega \leq C$. This together with the upper bound of u implies the closedness, where the bootstrap argument also uses the quasi-bounded geometry. This solves the Monge-Ampère type equation. Thus, we obtain a Kähler-Einstein metric ω_{KE} which is uniformly equivalent to ω. In particular, ω_{KE} is complete. The uniform estimates on u implies the curvature tensor of ω_{KE} has bounded covariant derivatives of any order. The uniqueness of ω_{KE} is already known, due to my Schwarz lemma. This completes the proof of Theorem 1.

This result can be generalized in several directions. Let me mentioned two of them.

Theorem 2 (Wu-Yau (2017))

Let (M, ω) be a complete Kähler manifold with bounded sectional curvature. Suppose that M has a holomorphic covering space $\tilde{M}$ such that for each point $x \in \tilde{M}$, there exists a holomorphic map F from $\tilde{M}$ to a Kähler manifold (N, ω_N) such that $H(\omega_N) \leq -1$ and

$$F^*\omega_N \geq C\tilde{\omega} \quad \text{at } x$$

where $\tilde{\omega}$ is the induced covering metric, and $C > 0$ is a constant independent of x. Then, M admits a complete Kähler-Einstein metric ω_{KE} which is uniformly equivalent to ω and the curvature tensor of ω_{KE} and all its covariant derivatives are bounded on M.

Theorem 2 contains the previous theorem (Theorem 1), in that if the complete metric ω on M has negatively pinched holomorphic sectional curvature, then in particular its sectional curvature is bounded. Furthermore, we can simple take $N = M$ and let F be the projection from its universal cover $\tilde{M}$ to M. The argument of Theorem 1 together with some covering space property implies Theorem 2.

Theorem 2 in particular implies the aforementioned result: If a compact Kähler manifold is Carathéodory hyperbolic, then its canonical bundle is ample.

Theorem 3 (Wu-Yau (2017))

Let (M, ω) be a complete Kähler manifold with bounded sectional curvature. Assume that M has a holomorphic covering space $\pi : \tilde{M} \to M$ satisfying the following conditions:

(i) *There exists a compact subset E of $\tilde{M}$ such that $dd^c \log \tilde{\omega}^n \geq C_1 \tilde{\omega}$ on $M \setminus E$, where C_1 is a constant and $\tilde{\omega} = \pi^* \omega$.*

(ii) *For each $x \in E$, there exists a holomorphic map F from $\tilde{M}$ to a Kähler manifold (N, ω_N) with $H(\omega_N) \leq -1$ such that $F^* \omega_N \geq C_2 \tilde{\omega}$ where C_2 is a constant independent of x.*

Then, M admits a complete Kähler-Einstein metric ω_{KE} which is uniformly equivalent to ω, and the curvature tensor of ω_{KE} has bounded covariant derivatives of arbitrary order.

A motivated example for Theorem 3 is the moduli space of Riemann surfaces, whose covering space is the Teichülcer space. The Bers embedding theorem exemplifies the map F from the covering space to a large ball in $\mathbb{C}^n$ so that the pullback metric under F is nondegenerate.

The complete Kähler-Einstein metric of negative scalar curvature is an example of the *invariant metric* on a complex manifold; that is, every biholomorphic map is an isometry relative to such a metric. Thus, an invariant metric depends only on the complex structure of the complex manifold. Besides the Kähler-Einstein metric, the classical invariant metrics also include the Bergman metric, the Carathéodory-Reiffen metric, and the Kobayashi-Royden metric.

We shall use the quasi-bounded geometry developed in the previous theorems to address several problems concerning the Kobayashi-Royden metric and Bergman metric.

The Kobayashi-Royden metric $\mathfrak{K}$ is the infinitesimal form of the Kobayashi pseudo-distance. Let M be a complex manifold. For each $x \in M$ and $X \in T'_x M$, consider a holomorphic map ϕ from the unit disk to M such that $\phi(0) = x$ and $\phi_*(v) = X$. The Kobayshi-Royden metric $\mathfrak{K}(x, X)$ is define to be the infimum of the Euclidean norm $|v|_0$ over all such maps ϕ.

A notable conjecture imposed by R. E. Greene and H. Wu (1979) asserts the following: If a complete, simply-connected, Kähler manifold has sectional curvature bounded between two negative constants, then its Kobayashi-Royden metric is uniformly equivalent to the background Kähler metric.

In fact, it is well-known that, due to the Schwarz Lemma, the Kobayashi-Royden metric is always bounded below by a hermitian metric, provided the holomorphic sectional curvature of the hermitian metric is bounded above by a negative constant. Thus, it is the upper bound of Kobayashi-Royden metric that requires a proof.

By using the quasi-bounded geometry and a comparison argument, Damin Wu and I can prove a stronger result, which removes the simply-connectedness, and relaxes the sectional curvature by the holomorphic sectional curvature.

Theorem 4 (Wu-Yau (2017))

If a complete Kähler manifold with holomorphic sectional curvature bounded between two negative constants, then its Kobayshi-Royden metric is uniformly equivalent to the background Kähler metric.

Let us now recall the Bergman metric. Let M be an n-dimensional complex manifold. There is a natural inner product on the smooth $(n, 0)$ forms given by

$$\langle \varphi, \psi \rangle = (-1)^{n^2/2} \int_M \varphi \wedge \overline{\psi}.$$

Notice that this inner product is independent of the hermitian metrics on M and on K_M.

Denote by $\mathcal{H}$ the set consisting of holomorphic n-forms φ such that the induced norm $\|\varphi\| < +\infty$. Then $\mathcal{H}$ is a separable Hilbert space. Assume $\mathcal{H} \neq \{0\}$. Then $\mathcal{H}$ contains an orthonormal basis $\{e_j\}_{j \geq 0}$ with respect to the inner product.

One can define an (n, n) form $\mathfrak{B}$ on $M \times M$ by

$$\mathfrak{B}(p, q) = \sum_{j \geq 0} e_j(p) \wedge \overline{e_j}(q).$$

This definition is independent of the choice of orthonormal basis. Along the diagonal of $M \times M$, we can express $\mathfrak{B}(p, p)$ in terms of local coordinates $(z^1, \ldots, z^n)$ as

$$\mathfrak{B}(z, z) = b(z, z) dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n.$$

We call $\mathfrak{B}(p, p)$ and $b(z, z)$ the *Bergman kernel form* and *Bergman kernel function* on M, respectively. When M is a domain in $\mathbb{C}^n$, the Bergman kernel function b recovers the classical Bergman kernel.

We further assume that the Bergman kernel form $\mathfrak{B} > 0$ everywhere on M. Let

$$\omega_{\mathfrak{B}} = dd^c \log b,$$

which is globally defined on M. We call $\omega_{\mathfrak{B}}$ the *Bergman metric* on M, if $dd^c \log b > 0$ everywhere on M.

By contrast to those Bergman metrics defined via a general positive line bundle, the Bergman metric given here is an invariant metric on M.

Based on my work with Siu in 1977, R. E. Greene and H. Wu (1979) proves the following result: If (M, ω) is complete, simply-connected, Kähler manifold whose sectional curvature is pinched between two negative constants, then M admits a Bergman metric $\omega_{\mathfrak{B}}$, which dominates ω, i.e., $\omega_{\mathfrak{B}} \geq C\omega$ on M. In particular, $\omega_{\mathfrak{B}}$ is complete.

Then, Greene-Wu (1979) conjectures that the Bergman metric $\omega_{\mathfrak{B}}$ is also dominated by ω; in other words, $\omega_{\mathfrak{B}}$ is uniformly equivalent to ω.

The topological constrain is needed here, as we have examples of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and the punctured disk.

Attempts

This conjecture of Greene-Wu would follow immediately if one can derive an volume estimate $\omega_{\mathfrak{B}}^n \leq C\omega^n$. However, the volume estimate is not easy, as the Schwarz lemma does not apply. The curvature of Bergman metric is unclear in general.

Greene-Wu (1979) proposes to show the following technical statement: For every $x \in M$, there is a uniform positive lower bound for $\|\varphi\|$ where φ runs over all square integrable holomorphic n-forms such that φ vanish at x of order 1. Indeed, such an estimate seems no easier than the conjecture itself, and is a consequence of our next result.

Damin Wu and I take a different approach, using the bounded geometry developed in the previous theorems to derive the pointwise interior estimate. This enables us to prove the conjecture of R. E. Greene and H. Wu.

Theorem 5 (Wu-Yau (2017))

If (M, ω) is a complete, simply-connected, Kähler manifold whose sectional curvature is bounded between two negative constants $-A$ and $-B$, then its Bergman metric $\omega_{\mathfrak{B}}$ has bounded geometry and satisfies

$$\omega_{\mathfrak{B}} \leq C\omega \quad \text{on } M,$$

where the constant $C > 0$ depends only on A, B, and $\dim M$. *Consequently, $\omega_{\mathfrak{B}}$ is uniformly equivalent to ω.*

As a consequence, on a complete, simply-connected, Kähler manifold M, the three classical invariant metrics, Kähler-Einstein, Bergman, and Kobayashi-Roydent metrics, are all uniformly equivalent. This proves an evidence of my conjecture that such a manifold should be biholomorphic to a bounded domain in $\mathbb{C}^n$.

At this end, let me mention another open problem. This is a question about resolution of singularities of Kähler metrics. Let us look at the following class of metrics: Take a complex variety M and a subvariety S of M, we consider Kähler metrics g defined in $M - S$ that satisfies the following condition: at each point $x \in S$, there is a neighborhood U of x so that a nonsingular manifold O and a subvariety D of O and a holomorphic map $F : O \to U$ which maps D into S so that each component of the inverse of $S \cap U$ is a compact subvariety of D. (In fact, a component of the inverse image of a compact neighborhood is compact.)

The map is locally invertible on every point in $O - D$, and the pullback of the metric g (defined on $M - S$) under F can be extended to be a smooth nonsingular metric on O. We also allow the pullback metric to be a Kähler metric defined on $O - D$, complete towards D and its curvature and covariant derivatives are bounded.

The Kähler metric g is said to *admit resolution of singularities* if a system of maps $\{F\}$ exists at every point $x \in S$. A good example is the orbifold metric where O can be taken to be the ball and the map F is the map from the ball to its quotient space which maps the origin to the quotient singularity. Note that the singular behavior of the metric depends on the system $\{F\}$ which is defined in holomorphic category.

I conjecture that if the curvature and the covariant derivatives of the curvature of this Kähler metric are bounded in each neighborhood of x of S, the resolution system $\{F\}$ exists. Such a statement may be called *resolution of singularities of Kähler metric*. Note that if we fix a holomorphic system $\{F\}$, there is only one complete Kähler-Einstein metric with negative Ricci curvature resolved by $\{F\}$.

On the other hand, there can be distinct Kähler-Einstein metrics if we choose different system to resolve the singularities of the metric. One can define two systems of resolutions to be equivalent if the holomorphic map from $O - D$ to $O' - D'$ can be extended to be a nonsingular map from O to O' and the same is true for the inverse map from $O' - D'$ to $O - D$.

This concept appeared in my works in the late 1970s with S.-Y. Cheng on the construction of Kähler-Einstein metrics on singular varieties. The existence of Kähler-Einstein metrics can be readily generalized to this class of singular metrics. (Basically the same argument I used with Cheng (1984).)

Is it true that algebraic manifolds of general type admits such Kähler metrics with negative Ricci curvature? It is certainly true for algebraic surface of general type. Since the arguments of Ricci flow largely depend only on maximal principle, Kähler-Ricci flow works well with class of singular metrics. Note that such Kähler metric includes a class of Kähler metrics which can be degenerate along S.

Thank you!