Plurisubharmonic envelopes and supersolutions

Joint work with H.C.Lu and A.Zeriahi

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- Berman-Guenancia (2014, semi-log canonical singularities).

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Goal of today's lecture : dictionary & applications.

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Theorem

If there exists a subsolution, then there exists a unique solution which is

- *bounded on X and continuous in $\mathrm{Amp}(\omega)$;*
- *the quasi-psh projection of the lower envelope of supersolutions.*

Classical sub/super/solutions

Definition

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PROBLEM: classical solutions usually do not exist !

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- Main difficulty : passing from *continuous* to *bounded*.

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- Pluripot. solutions are quasi-continuous but not nec. continuous.

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- Local Berman approximation process.

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A **viscosity solution** *of (CMA) is a bounded function which is both a viscosity subsolution and a viscosity supersolution.*

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Theorem (EGZ11)

$\varphi : X \to \mathbb{R}$ is a viscosity subsolution iff it is a pluripotential subsolution.

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Theorem (GLZ17)

Assume $\varphi : X \to \mathbb{R}$ is a viscosity supersolution of (CMA). Then $P(\varphi)$ is a pluripotential supersolution, where

Envelopes and supersolutions

It follows from the pluripotential comparison principle that

$$\varphi \text{ pluripotential supersolution } \Rightarrow \varphi_* \text{ viscosity supersolution }.$$

Conversely we have the following more delicate implication :

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(viscosity subsolution $\Rightarrow$ ω-psh $\Rightarrow$ quasi-continuous, etc.)

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- $-\mathbf{1}_E$ is not a viscosity supersolution of $(\omega + dd^c u)^n_+ \leq C\omega^n$.

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- Alternative approach in [GLZ17] : lower envelopes of supersolutions.

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Of course the truth is somewhat more complicated...

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- Our proof is again through Berman approximation process.

The end

Thank you for your attention !