Curvature of (higher) direct images.

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Theorem

Assume that $\Omega \geq 0$. Then there is a holomorphic vector bundle, E, over B with fibers

$$E_t = H^{n,0}(X_t, L_t).$$

We equip E_t with the L^2-metric

$$\|u\|_t^2 = c_n \int_{X_t} u \wedge \bar{u} e^{-\phi}.$$

Then $(E, \|\cdot\|)$ has semipositive curvature (in the sense of Nakano).

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$$c(\Omega) = \frac{\Omega^{n+1}/(n+1)}{\Omega^n \wedge idt \wedge d\bar{t}},$$

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where t is a local coordinate on B.

Second, there is a $(1,0)$- vector field, V_ϕ, on $\mathcal{X}$, depending on ϕ, such that $dp(V_\phi) = \frac{\partial}{\partial t}$ (V_ϕ is a *lift* of $\frac{\partial}{\partial t}$), with the property that $\Omega(V_\phi, \bar{W}) = 0$ if W is a vertical field (V_ϕ is *horisontal*, see Siu-Schumacher).

Let $\kappa_\phi = \bar{\partial}_{X_t} V_\phi \in \mathbb{Z}^{0,1}(T^{1,0}(X_t))$. It is a very special representative of the Kodaira-Spencer cohomology class $\kappa \in H^{0,1}(X_t, T^{1,0})$.

The form κ_ϕ acts on u in $H^{n,0}(X_t, L_t)$ by contraction and wedging. The result is written $\kappa_\phi \cup u$.

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$$\langle \Theta_{\frac{\partial}{\partial t}, \frac{\partial}{\partial \bar{t}}} u, u \rangle_t = \langle c(\Omega) u, u \rangle_t + \langle (1 + \Box)^{-1} \kappa_\phi \cup u, \kappa_\phi \cup u \rangle_t$$

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If the curvature is zero, we must have that ϕ solves the HCMA equation $(i\partial\bar{\partial}\phi)^{n+1} = 0$, and moreover $\kappa_\phi = 0$

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(When $B = \{t \in \mathbb{C}; 0 < \mathrm{Re}\, t < 1\}$ and $\phi_t = \phi_{\mathrm{Re}\, t}$, $c(\Omega)$ is the geodesic curvature of the curve $t \to \phi_t$ in the Mabuchi space.)

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Example: Take $n = 1$, $L_t = K_{X_t}$, i e $L = K_{\mathcal{X}/B}$. Assume $L_t > 0$, so that all fibers are compact Riemann surfaces of genus at least 2. Then

$$H^{0,n}(X_t, -L_t) = H^{0,1}(X_t, T^{1,0}),$$

which is where KS-classes live. We can take ϕ_t to be the potential of the Kähler-Einstein metric on X_t. $\|u\|^2$ is the Weil-Peterson norm of u. So we get a formula for the curvature of the WP-metric.(Wolpert -84) $\square$

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If we combine the theorem with a theorem of Schumacher which says that $c(\Omega) \geq 0$, we also find that the curvature of the WP-metric is negative (Ahlfors).

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We can make this fit the previous discussion by looking at the bundle $E^{n,0}$ with fibers $H^{n,0}(X_t, -K_{X_t}) = \mathbb{C}$. This is a trivial bundle with $u^0 = 1$ as trivializing section. Take $\kappa \in H^{0,1}(X_t, T^{1,0}(X_t))$ and associate to κ, $u^1 := \kappa \cup u^0 \in H^{n-1,1}(X_t, -K_{X_t})$. Then $\|\kappa\| = \|u^1\|$, so we can interpret Siu's formula as a curvature formula for $E^{n-1,1}$.

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More generally we can look at the bundles $E^{p,q}$ with fibers $H^{p,q}(X_t, -K_{X_t})$, endowed with the L^2-metric induced by the KE-metric, for $p + q = n$. Schumacher (-10) generalized Siu's formula to all $E^{p,q}$, $p + q = n$.

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In joint work with M Paun and X Wang we generalized this as follows:

Theorem

Let $\mathcal{X} \to B$ be a smooth proper fibration, and let (ϕ, L) be a hermitian line bundle over $\mathcal{X}$ with $\Omega = i\partial\bar{\partial}\phi > 0$ on fibers. Fix (p, q) with $p + q = n$ and assume that all the spaces $H^{p,q}(X_t, -L_t)$ have the same rank. Then they form a vector bundle $E^{p,q}$. Give $E^{p,q}$ the L^2-metric induced by ϕ and $\Omega|_{X_t}$. Let Θ be the curvature of the Chern connection on this hermitian bundle. Then

$$\langle \Theta_{\frac{\partial}{\partial t}, \frac{\partial}{\partial \bar{t}}} u, u\rangle = -\langle (1+\Box)^{-1}\mu_\perp, \mu_\perp\rangle - \langle (1+\Box)^{-1}\xi, \xi\rangle - \langle c(\Omega)u, u\rangle + \|\eta_h\|^2,$$

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The same formula was obtained independently with a different proof by Ph Naumann.

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Lemma

Let X compact, $(\phi, L) \to X$ given with $i\partial\bar\partial\phi > 0$. Let u be a (p,q)-form with values in $-L$, with $p + q = n$ and u harmonic for $\phi, \omega_\phi = i\partial\bar\partial\phi$. Then u is primitive, i e $\omega_\phi \wedge u = 0$.

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$$\|u\|^2 = c_{p,q}\int_{X_t} u\wedge \bar u e^{-\phi} = c_{p,q} p_*(u\wedge\bar u e^{-\phi}).$$

Therefore the same principle of proof as in the $(n,0)$-case applies (with some additional complications).

Recall the formula again:

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Remarks:

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4. When $(p, q) = (0, n)$, the curvature is negative (which we already knew).

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6. Let $\phi_q = \log \|u^q\|^2$. Then it follows from the formula that

$$i\partial\bar{\partial}\phi_q \geq e^{\phi_q - \phi_{q-1}} - e^{\phi_{q+1} - \phi_q},$$

for $1 \leq q \leq n$, where $\phi_{n+1} := -\infty$.

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7. From this one sees that for suitably chosen $a_j > 0$,

$$\sum a_j \phi_j$$

defines a metric on B with curvature bounded from above by a negative constant (variation of argument by To-Yeung and Schumacher). In particular B can not be $\mathbb{C}$. So, if $\mathcal{X} \to Y$ is a fibration as before, Y cannot contain an entire curve, i e Y is Brody hyperbolic.

There is also a variant of the curvature formula for the case when the metric ϕ is not strictly positive, but instead flat on fibers, $i\partial\bar{\partial}\phi|_{X_t} = 0$. Then the curvature form of ϕ is the pullback of a form c on the base.

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In this case we get the curvature formula for the $E^{p,q}$-bundles

$$\langle \Theta_{\frac{\partial}{\partial t}, \frac{\partial}{\partial \bar{t}}} u, u \rangle = c_{\frac{\partial}{\partial t}, \frac{\partial}{\partial \bar{t}}} \|u\|^2 + \|\xi_h\|^2 - \|\eta_h\|^2.$$

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$$\langle \Theta_{\frac{\partial}{\partial t}, \frac{\partial}{\partial \bar{t}}} u, u \rangle = c_{\frac{\partial}{\partial t}, \frac{\partial}{\partial \bar{t}}} \|u\|^2 + \|\xi_h\|^2 - \|\eta_h\|^2.$$

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An example is when $L = K_{\mathcal{X}/B}$ and the fibers are Calabi-Yau. Then we can take ϕ to be the Bergman kernel metric, and get a picture very similar to the canonically polarized case.

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This suggests yet another extension, to the case when the map p is not a smooth fibration, but just a surjective map, and may have singular fibers over a Zariski closed set.

In this situation we let Δ be the (codimension 1-components of) the set of singular values, and let $W = p^{-1}(\Delta)$. We then consider instead the sheaves

$$\mathcal{R}^q p_*\big(\Omega^{n-q}_{\mathcal{X}/B} < W > \otimes(-L)\big)$$

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We are then able to extend our metrics to these sheaves (modulo torsion), provided we have information about the curvature of L. This requires an analysis of the behaviour of our metric on $-L$ near the singular fibers, and this is where the advantage of working with a more general metric than the KE-metric appears; by algebraic-geometric methods we construct another metric which can be estimated.

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This metric has the additional advantage of having stronger positivity properties.

The net result of all this is a metric version of the following theorem of Viehweg-Zhuo.

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Theorem

Let $p : \mathcal{X} \to Y$ be a family of canonically polarized manifolds of maximal variation. Then there exists a positive $q \leq n$ such that the bundle $Sym^q(\Omega_Y < \Delta >)$ contains a (non trivial) big coherent subsheaf.

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The subsheaf is essentially the dual of the kernels of the iterated Kodaira-Spencer maps - for which we have a strict negative bound of the curvature.

Thanks!