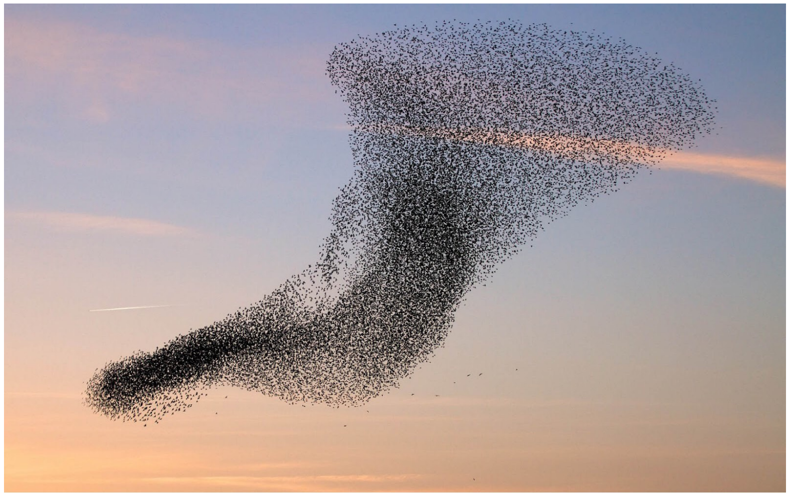
# Canonical metrics, random point processes and tropicalization

# (Grenoble, 2017)



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## Outline

Survey of the probabilistic construction of canonical metrics

1. $X$ is variety of positive Kodaira dimension

2. $X$ is Fano (conjectural picture)

3. $X$ is toric and tropicalization (supporting the conjectural picture)

**Motivation**

Let $X$ be a $n-$dim. complex projective algebraic variety and consider the case when $K_X > 0$ and non-singular.

By the Aubin-Yau theorem $X$ admits a unique Kähler-Einstein $\omega_{KE}$ metric with negative Ricci curvature:

$$\text{Ric}\omega_{KE} = -\omega_{KE} \tag{1}$$

In particular, there is a *canonical* (normalized) volume form $dV_{KE}$ on $X$ :

$$dV_{KE} := \omega_{KE}^n / V$$

Conversely, we can recover the metric $\omega_{KL}$ from the volume form $dV_{KE}$ :

$$i\partial\bar{\partial}(\log dV_{KE}) = \omega_{KE}$$

Hence, the fundamental canonical object attached to $X$ is a *volume form*.

**Varities of positive Kodaira dimension**

More generally, Song-Tian and Tsuji introduced a *canonical measure* $\mu_X$ on any variety $X$ with positive Kodaira dimension, i.e. when

$$\dim H^0(X, kK_X) \to \infty, \quad k \to \infty$$

where $H^0(X, kK_X)$ is the space of all pluricanonical forms at level $k$.

- The measure $\mu_X$ is birationally invariant

## The probabilistic point of view

Is there a canonical way of sampling $N$ points on $X$ at random so that the canonical measure $\mu_X$ emerges as $N \to \infty$?

- Of course, we could always sample wrt $\mu_X$, but this is cheating as we want to *recover* $\mu_X$ from this process!

- More specifically, we want the sampling procedure to be "algebraic" i.e. encoded by the canonical ring of $X$ :

$$\bigoplus_{k\geq 1} H^0(X, kK_X)$$

- In accordance with the usual philosophy in Kähler geometry

## The definition of the canonical probability measures

Setting

$$N_k := \dim H^0(X, kK_X)$$

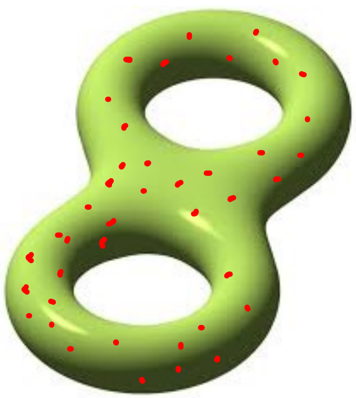
we will define a sequence of probability measures $\mu^{(N_k)}$ on $X^{N_k}$ which are symmetric, i.e. invariant under the action of the permutation group $\Sigma_{N_k}$.

Then $\mu^{(N_k)}$ descends to define a probability measures on the space

$$X^{(N)} := X^N / \Sigma_N$$

of configurations of $N_k$ points on $X$.

- By definition this is a *random point process* on $X$ with $N_k$ particles



To define a probability measure $\mu^{(N_k)}$ on $X^{N_k}$ we just need to find a "canonical" element

$$S^{(k)} \in H^0(X^{N_k}, kK_{X^{N_k}})$$

We can then define

$$\mu^{(N_k)} := \frac{1}{Z_{N_k}} \left| S^{(k)}(z_1, ..., z_{N_k}) \right|^{2/k} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{N_k} \wedge d\bar{z}_{N_k},$$

where $Z_{N_k}$ is the normalizing constant ensuring that $\mu^{(N_k)}$ is a probability measure. Note however:

- We want $\left| S^{(k)}(z_1, ..., z_{N_k}) \right|$ to by symmetric (i.e. permutation invariant)

- It is enough if $S^{(k)}$ is determined up to a multiplicative complex number

This can be achieved by taking $S^{(k)}$ to be in the totally anti-symmetric part of

$$H^0(X^{N_k}, kK_{X^{N_k}})(= H^0(X, kK_X) \otimes \cdots \otimes H^0(X, kK_X), \quad N_k \text{ times})$$

i.e. $S^{(k)}$ is a generator of the determinant line

$$\Lambda^{N_k} H^0(X, kK_X) \subset H^0(X^{N_k}, kK_{X^{N_k}})$$

Concretely, taking a basis $s_i^{(k)}$ in $H^0(X, kK_X)$ we can take

$$S^{(k)}(x_1, x_2, ...x_{N_k}) := \det \left( s_i^{(k)}(x_j) \right)_{1 \leq i,j \leq N_k}$$

Accordingly, we will denote a generator of $\Lambda^{N_k} H^0(X, kK_X)$ by $\det S^{(k)}$ which is thus a section of $kK_{X^{N_k}} \to X^N$

It is easy to see that the probability measures are birationally invariant

**Main theorem (B. 13)**

Let $X$ be a projective variety with positive Kodaira dimension. Then the empirical measures of the canonical random point process converge in law towards the Song-Tian-Tsuji canonical measure $\mu_X$ on $X$ :

$$\frac{1}{N_k}\sum_{i=1}^{N_k}\delta_{x_i}\to\mu_X,$$

Concretely, when $K_X > 0$ this implies the the weak convergence of the following canonical volume forms on $X$ :

$$\nu_N := \int_{X^{N-1}} \mu^{(N)} \to dV_{KE},$$

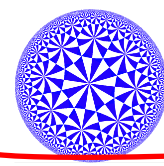
as $N \to \infty$

**Corollary**

Assume that $K_X > 0$. Then the following "Bergman type" Kähler metrics

$$\omega_k := i\partial\bar{\partial} \log \int_{X^{N_k-1}} \left|(\det S^{(k)})(\cdot, z_1 \ldots, z_{N_k-1})\right|^{2/k} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{N_k-1} \wedge d\bar{z}_{N_k-1}$$

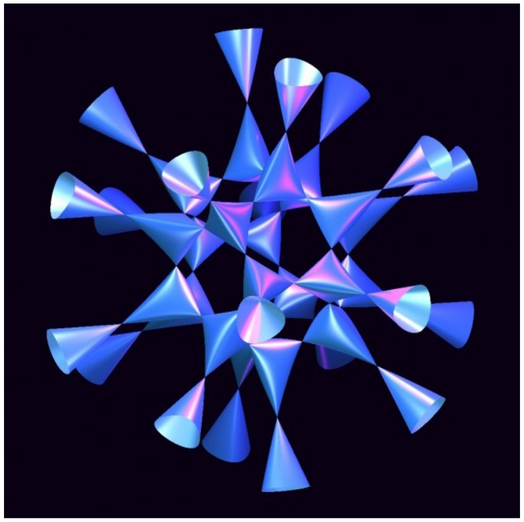
converge weakly on $X$ towards the unique Kähler-Einstein metric $\omega_{KE}$.



- More generally, when $X$ is of general type, the current $\omega_k$ is singular along the base locus of $kK_X$ and $\omega_{KE}$ is the singular Kähler-Einstein metric [Eyssidieux-Guedj-Zeriahi,...].

- When $K_X > 0$ this is somewhat similar to the convergence of balanced Bergman metrics [Donaldson, ...]

- But one virtue here is that it also works in singular and degenerate settings.



In fact, the proof of the main theorem shows that the convergence of the empirical measure

$$\delta_N := \frac{1}{N_k} \sum_{i=1}^{N_k} \delta_{x_i}$$

towards the deterministic meaure $\mu_X$ is *exponential* in the sense of *large deviations.*

The corresponding rate functional $F(\mu)$ has the property that

$$F(\omega^n/V)$$

coincides with Mabuchi's K-energy $M(\omega)$ of $\omega$ (which is minimized on $\omega_{KE}$)

## The Fano case

Now consider the opposite setting when $X$ is a Fano manifold, i.e.

$$-K_X > 0$$

A KE-metric $\omega_{KE}$ must have *positive* Ricci curvature.

**Yau-Tian-Donaldson conjecture:** $X$ admits a KE metric iff $X$ is K-polystable

[settled by Chen-Donaldson-Sun, for $X$ smooth]

An alternative variational approach gives [B.-Boucksom-Jonsson]:

- A Fano manifold $X$ admits a unique KE metric iff $X$ is uniformly K-stable

## The probabilistic approach

Recall that in the case $K_X > 0$, the probability measure on $X^{N_k}$ is defined by

$$\mu^{(N_k)} := \frac{1}{Z_{N_k}} \left| S^{(k)}(z_1, ..., z_{N_k}) \right|^{2/k} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{N_k} \wedge d\bar{z}_{N_k},$$

where $\Psi^{(k)}$ is built from the $N_k-$dimensional vector space $H(X, kK_X)$

- However, when $-K_X > 0$ the spaces $H^0(X, -kK_X)$ are trivial!
- Instead, we need to work with the spaces $H^0(X, -kK_X)$
- We are then forced to replace the power $2/k$ with $-2/k$

On a Fano variety $X$ it is thus natural to try to define

$$\mu^{(N_k)} := \frac{1}{Z_{N_k}} \left| S^{(k)}(z_1, ..., z_{N_k}) \right|^{-2/k} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{N_k} \wedge d\bar{z}_{N_k},$$

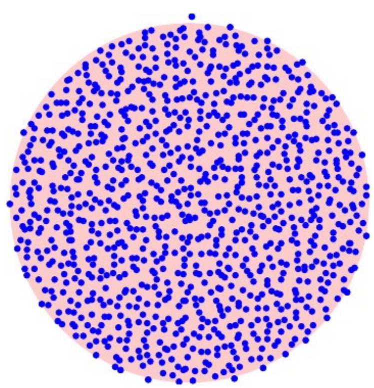
where $S^{(k)}$ is a generator of the determinant of $H^0(X, -kK_X)$.

- However, now the normalizing constant $Z_{N_k}$ may diverge!

**Definition:** A Fano manifold $X$ is said to be *Gibbs stable* if $Z_{N_k} < \infty$, for $k >> 1$.

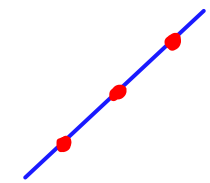
**Conjecture:** Let $X$ be a Fano manifold. Then

- $X$ is Gibbs stable iff $X$ admits a unique KE-metric $\omega_{KE}$
- In that case, $\omega_{KE}$ emerges in the many particle limit of the corresponding random point processes



## Gibbs stability

The notion of Gibbs stability admits a natural algebro-geometric formulation:



Consider the following divisor $\mathcal{D}_k$ on $X^N$,

$$\mathcal{D}_k = (S^{(k)} = 0) = \{(x_1, .., , x_{N_k}) : \ \exists s_k \in H^0(X^{N_k}, -kK_{X_{N_k}}) : s_k(x_i) = 0\}$$

- $\mathcal{D}_k/k$ is a $\mathbb{Q}-$divisor on $X^{N_k}$ such that $\mathcal{D}_k/k \sim -K_{X^{N_k}}$
- Gibbs stability of a Fano $X$ at level $k$ means that $\mathcal{D}_k/k$ has mild singularities in the sense of the MMP.

(klt)

In other words, $X$ is Gibbs stable iff

$$\mathrm{lct}(X^{N_k}, \frac{1}{k}\mathcal{D}_k) > 1, \;\; k >> 1$$

This also suggests a stronger notion of Gibbs stability *"uniform Gibbs stability"*:

$$\liminf_{k\to\infty} \;\; \mathrm{lct}\; (X^{N_k}, \tfrac{1}{k}\mathcal{D}_k) > 1$$

Using MMP techniques Fujita and Fujita-Odaka have recently shown:

$$\text{uniform Gibbs stability} \implies \text{uniform K-stability}$$

Hence, by the resolution of the YTD

$$\text{uniform Gibbs stability} \implies \text{existence of } \omega_{KE}$$

But the convergence of the point processes is still open (and the converse implication)

**The case when Aut $(X)_0$ is non-trivial**

If Aut $(X)_0$ is non-trivial, then $X$ is *not* Gibbs stable:

$$Z_{N_k} = \infty$$

We have to break the symmetry!

Fix a volume form $dV$ on $X$. It induces a metric $\|\cdot\|$ on $-K_X$.

For any sufficently small $\gamma > 0$ the "regularized" probability measure on $X^{N_k}$

$$\mu_{-\gamma}^{(N)} := \frac{1}{Z_{N_k,-\gamma}} \left\| S^{(k)} \right\|^{-2\gamma/k} dV^{\otimes N_k}$$

is well-defined when $k >> 1$.

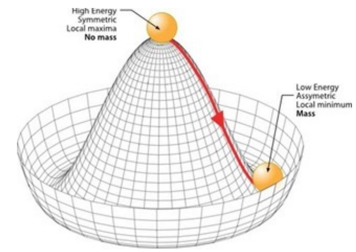
More precisely, it is well-defined as long as $\gamma < \gamma(X)$, where the *critical exponent* $\gamma(X)$ is given by

$$\gamma(X) := \liminf_{k\to\infty} \ \text{lct}\ (X^{N_k}, \tfrac{1}{k}\mathcal{D}_k)$$

## Conjecture

- The critical exponent $\gamma(X)$ coincides with the sup over all $\gamma$ such that Aubin's equation

$$\text{Ric } \omega = \gamma\omega + (1-\gamma)\text{Ric } dV$$




admits a solution $\omega_\gamma$ (assumed "minimizing" if $\gamma > 1$)

- If $\gamma < \gamma(X) \leq 1$, then the unique solution $\omega_\gamma$ emerges in the many particle limit of the corresponding random point process.

- If $X$ is "Gibbs polystable", then one gets a Kähler-Einstein metric $\omega$ by first letting $N \to \infty$ and then increasing $\gamma \to 1$ [to de defined...]

**Relations to statistical mechanics and proof stategy**

Fixing a volume form $dV$ on $X$ and $\beta \in \mathbb{R}$ we consider the probability measure

$$\mu_\beta^{(N)} := \frac{1}{Z_{N_k,\beta}} \left\| S^{(k)} \right\|^{2\beta/k} dV^{\otimes N_k} \text{ on } X^{N_k}$$

This gives the *canonical* probability measures when $\beta = \pm 1$.

We can rewrite

$$\mu_\beta^{(N)} := \frac{1}{Z_{N_k,\beta}} e^{-\beta E^{(N)}(x_1,\ldots,x_N)} dV^{\otimes N_k} \text{ on } X^{N_k},$$

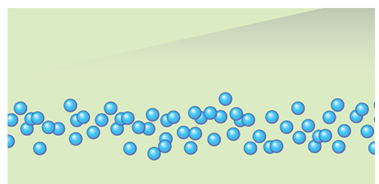
where

$$E^{(N)} := -k^{-1} \log \left\| S^{(k)} \right\|^2$$

In general, a probability measure of the form

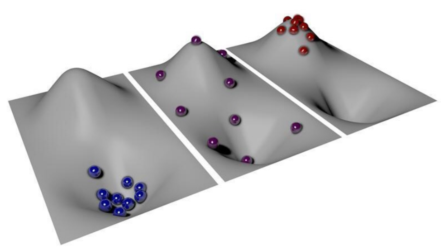
$$\mu_\beta^{(N)} := \frac{1}{Z_{N,\beta}} e^{-\beta E^{(N)}(x_1,\ldots,x_N)} dV^{\otimes N_k} \text{ on } X^N,$$

is the *Gibbs measure* describing the equilibrium distribution of $N$ particles on $X$ with interaction energy $E^{(N)}$ at inverse temperature $\beta$.



- In our case the interaction $E^{(N)}(x_1, \ldots, x_N) = -k^{-1} \log \left\| S^{(k)} \right\|^2$ is *repulsive*

- The case when $\beta = -\gamma < 0$ equivalently corresponds to an *attractive* interaction at inverse temperature $\gamma$.



The starting point of the proof of the convergence when $\beta > 0$ is that

$$\frac{1}{N_k}\sum_{i=1}^{N_k}\delta_{x_i}\to\mu \implies N^{-1}E^{(N)}(x_1,...,x_N)\to E(\mu),$$

in the sense of $\Gamma-$convergence [using B.-Boucksom'11]

- $E(\mu)$ is the *pluricomplex energy* of $\mu$ (i.e. $E(\mu) = (I-J)(\varphi_\mu)$)

Heuristically, this suggests the following asymptotics:


$$\mathbb{P}\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}\in B_\epsilon(\mu)\right) := \frac{1}{Z_N}\int_{B_\epsilon(\mu)} e^{-\beta E^{(N)}} dV^{\otimes N}$$

$$\sim e^{-NF_\beta(\mu)}, \ , N\to\infty,\ \epsilon\to 0,$$

where the "rate functional" $F_\beta$ is a lsc functional on $\mathcal{P}(X)$ :

$$F_\beta(\mu) = \beta E(\mu) + \text{Ent}_{\,dV}(\mu) - C_\beta$$

- This is classical when $\beta = 0$ (i.e. $E = 0$) [Boltzmann, Sanov,...]

- Thus, the probability is exponentially small, unless $F_\beta(\mu) = 0$

- $F_\beta(\omega^n) = M(\omega)$ :=Mabuchi's K-energy. So this "explains" the Chen-Tian energy+entropy formula for $M(\omega)$

- The actual proof of the asymptotics exploits that $E^{(N)}$ is uniformly *quasi-superharmonic* on the orbifold $X^N/\Sigma_N$.

- To handle the case $\beta < 0$ one should exploit the *pluri*subharmonicity...

**Supporting evidence in the Fano case** $(\beta = -1)$

- The conjectures do hold when $X$ is a *one dimensional* log Fano.


- Then Gibbs stability is equivalent to uniform Gibbs stability and K-stability

- If $X$ admits a KE metric, then the functional $F_{-1}(\mu)$ is lsc on $\mathcal{P}(X)$ [BBGEZ]

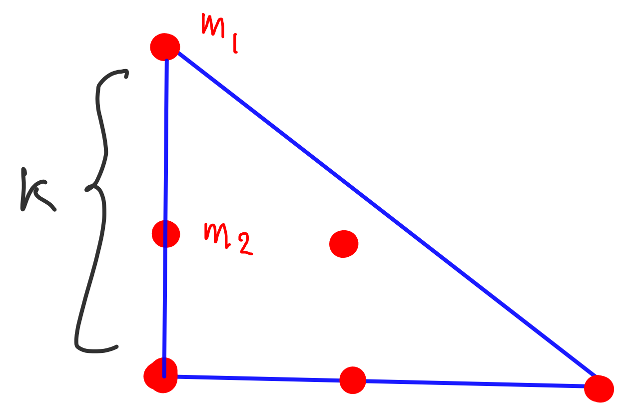
## The toric case and tropicalization

Let now $X$ be a toric Fano variety, i.e. $X$ is an equivariant compactification of the complex torus $\mathbb{C}^{*n}$.

By the usual toric dictionary

$$(X, -K_X) \longleftrightarrow P,$$

where $P$ is a certain polytope $P \subset \mathbb{R}^n$.




- Let $m_1, ..., m_{N_k}$ be an enumaration of the integer points of $kP$
- The corresponding multinomials $z^{m_i}$ span $H^0(X, -kK_X)$

Accordingly, we can write the *non-normalized* measure on $(\mathbb{C}^{*n})^N$ as

$$|\Delta(z_1, ... z_{N_k})|^{-2/k} \left( \frac{dz}{z} \wedge \frac{\overline{dz}}{z} \right)^{\otimes N_k}$$

where

$$\Delta(z_1, ... z_{N_k}) = \sum_{\sigma \in S_N} (-1)^{\text{sign}(\sigma)} z_1^{m_{\sigma(1)}} \dots z_{N_k}^{m_{\sigma(N)}} \quad (2)$$

## Exploiting the torus symmetry

Recall that, modulo **Aut** $(X)_0$ any Kähler-Einstein metric $\omega_{KE}$ on a toric variety $X$ is invariant under the action of the *real* torus $T$.

In particular, its volume form $dV_{KE}$ on $\mathbb{C}^{*n}$ "descends" to $\mathbb{R}^n$ under

$$\text{Log}: \ \mathbb{C}^{*n} \to \mathbb{R}^n, \ \ z \mapsto x := (\log|z_1|^2, ..., \log|z_n|^2),$$

$$dV_{KE} = e^{-\psi(z)}\frac{dz}{z} = e^{-\phi(x)}dx \wedge dy,$$

where $dy$ is the invariant top form on the torus fiber $T$

$$\psi(z) \text{ psh} \longleftrightarrow \phi(x) \text{ convex}$$

However, the measure

$$|\Delta(z_1, ...z_{N_k})|^{2/k} \left( \frac{dz}{z} \wedge \overline{\frac{dz}{z}} \right)^{\otimes N_k}$$

does not "descend" to $(\mathbb{R}^n)^N$, i.e.

$$E(z_1, ...z_{N_k}) := \log |\Delta(z_1, ...z_{N_k})|^2$$

is not $T^N-$invariant!

## Tropicalization

*The philosphy of Tropicalization:* replace an elusive problem for polynomials in $\mathbb{C}^d$ with a simpler one, for piece-wise affine convex functions in $\mathbb{R}^d$ :

$$\sum_{\boldsymbol{m}\in P} c_{\boldsymbol{m}} \boldsymbol{z}^{\boldsymbol{m}} \rightsquigarrow \phi(\boldsymbol{x}) := \max_{\boldsymbol{m}\in P} \boldsymbol{x}\cdot\boldsymbol{m}$$

Equivalently, the psh function on $\mathbb{C}^d$

$$\Psi(\boldsymbol{z}) := \log|P(\boldsymbol{z})|$$

is replaced by a convex function on $\mathbb{R}^d$ :

$$\phi(\boldsymbol{x}) := \lim_{\lambda\to\infty} \lambda^{-1}\Psi(e^{\lambda \boldsymbol{x}}, e^{iy})$$

Applying this philosphy here suggests studying the point processes on $\mathbb{R}^n$ defined by the following measure on $(\mathbb{R}^n)^N$ :

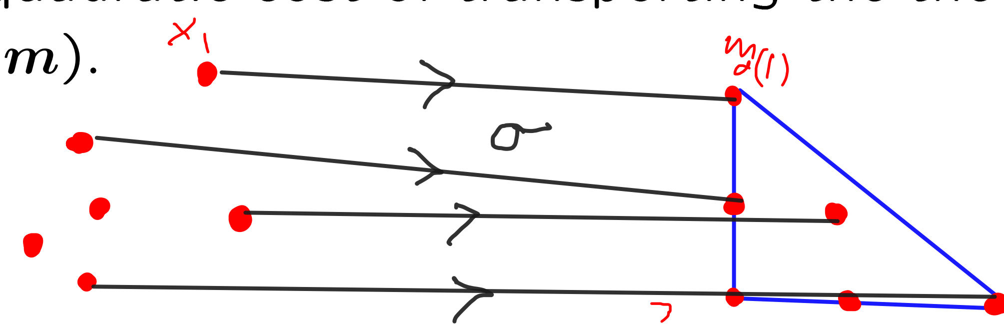
$$e^{-E_{trop}(x_1,\ldots,x_N)}dx^{\otimes N},$$

where

$$E_{trop}(x_1, \ldots, x_N) := \max_{\sigma \in S_N} \left( x_1 \cdot m_{\sigma(1)} + \ldots + x_N \cdot m_{\sigma(N)} \right),$$

$m_1, \ldots, m_N$ are the lattice points in $kP$.

- *Optimal transport interpretation:* $-E_{trop}(x_1, \ldots, x_N)$ is the [redacted]al minimal quadratic cost of transporting the the measure $\delta_N(\boldsymbol{x})$ to $\delta_N(\boldsymbol{m})$.



The action of $T_{\mathbb{C}} \Subset$ **Aut** $(X)_0$ on $X$ corresponds to the translation action on $\mathbb{R}^n$.

It forces

$$Z_N := \int e^{-E_{trop}(x_1,...,x_N)} dx^{\otimes N} = \infty$$

for any polytope $P$.

Again, we have to break the symmetry: $E_{trop}(x_1, ..., x_N) \rightsquigarrow$

$$\gamma E_{trop}(x_1, ..., x_N) + (1 - \gamma)\left(\phi_P(x_1) + ... + \phi_P(x_N)\right)$$

Then $Z_{N,\gamma}$ is finite for $\gamma < \gamma_P$

**Thm** (B.- Önnheim):

- The critical exponent $\gamma_P$ coincides with the invariant $R(X_P)$
- When $\gamma < \gamma_P$ the unique solution $\omega_\gamma$ of the Aubin-Yau equation emerges in the many particle limit of the corresponding random point process.
- When $X_P$ is K-polystable one gets a KE metric $\omega_{KE}$ by first letting $N \to \infty$ and then $\gamma \to 1$.

The proof exploits convexity of $E_{trop}$, i.e. that the corresponding probability measures are log-concave (Prekopa inequality, Borell's inequality, ...).

# Thank you!!!