Canonical metrics, random point processes and tropicalization

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Outline

Survey of the probabilistic construction of canonical metrics

1. $X$ is variety of positive Kodaira dimension

2. $X$ is Fano (conjectural picture)

3. $X$ is toric and tropicalization (supporting the conjectural picture)
Motivation

Let $X$ be a $n$–dim. complex projective algebraic variety and consider the case when $K_X > 0$ and non-singular.

By the Aubin-Yau theorem $X$ admits a unique Kähler-Einstein $\omega_{KE}$ metric with negative Ricci curvature:

$$\text{Ric}_{\omega_{KE}} = -\omega_{KE}$$  \hspace{1cm} (1)

In particular, there is a \textit{canonical} (normalized) volume form $dV_{KE}$ on $X$:

$$dV_{KE} := \omega_{KE}^n / V$$
Conversely, we can recover the metric $\omega_{KL}$ from the volume form $dV_{KE}$:

\[ i\partial\bar{\partial} (\log dV_{KE}) = \omega_{KE} \]
Hence, the fundamental canonical object attached to $X$ is a volume form.
Varities of positive Kodaira dimension

More generally, Song-Tian and Tsuji introduced a \textit{canonical measure} $\mu_X$ on any variety $X$ with positive Kodaira dimension, i.e. when

\[
\dim H^0(X, kK_X) \to \infty, \quad k \to \infty
\]

where $H^0(X, kK_X)$ is the space of all pluricanonical forms at level $k$.

- The measure $\mu_X$ is birationally invariant
The probabilistic point of view

Is there a canonical way of sampling $N$ points on $X$ at random so that the canonical measure $\mu_X$ emerges as $N \to \infty$?

- Of course, we could always sample wrt $\mu_X$, but this is cheating as we want to recover $\mu_X$ from this process!

- More specifically, we want the sampling procedure to be "algebraic" i.e. encoded by the canonical ring of $X$:

$$
\bigoplus_{k \geq 1} H^0(X, kK_X)
$$

- In accordance with the usual philosophy in Kähler geometry
The definition of the canonical probability measures

Setting

\[ N_k := \dim H^0(X, kK_X) \]

we will define a sequence of probability measures \( \mu^{(N_k)} \) on \( X^{N_k} \) which are symmetric, i.e. invariant under the action of the permutation group \( \Sigma_{N_k} \).
Then $\mu^{(N_k)}$ descends to define a probability measures on the space

$$X^{(N)} := X^N / \Sigma_N$$

of configurations of $N_k$ points on $X$.

- By definition this is a random point process on $X$ with $N_k$ particles
To define a probability measure $\mu^{(N_k)}$ on $X^{N_k}$ we just need to find a “canonical” element

$$S^{(k)} \in H^0(X^{N_k}, kK_{X^{N_k}})$$
We can then define

\[ \mu^{(N_k)} := \frac{1}{Z_{N_k}} \left| S^{(k)}(z_1, \ldots, z_{N_k}) \right|^{2/k} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{N_k} \wedge d\bar{z}_{N_k}, \]

where \( Z_{N_k} \) is the normalizing constant ensuring that \( \mu^{(N_k)} \) is a probability measure. Note however:

- We want \( \left| S^{(k)}(z_1, \ldots, z_{N_k}) \right| \) to be symmetric (i.e. permutation invariant)

- It is enough if \( S^{(k)} \) is determined up to a multiplicative complex number
This can be achieved by taking $S^{(k)}$ to be in the totally anti-symmetric part of

$$H^0(X^{N_k}, kK_{X^{N_k}}) = H^0(X, kK_X) \otimes \cdots \otimes H^0(X, kK_X), \text{ } N_k \text{ times}$$

i.e. $S^{(k)}$ is a generator of the determinant line

$$\Lambda^{N_k} H^0(X, kK_X) \subset H^0(X^{N_k}, kK_{X^{N_k}})$$

Concretely, taking a basis $s^{(k)}_i$ in $H^0(X, kK_X)$ we can take

$$S^{(k)}(x_1, x_2, \ldots x_{N_k}) := \det \left( s^{(k)}_i(x_j) \right)_{1 \leq i, j \leq N_k}$$

Accordingly, we will denote a generator of $\Lambda^{N_k} H^0(X, kK_X)$ by

$$(\det) S^{(k)}$$

which is thus a section of $kK_{X^{N_k}} \to X^N$
It is easy to see that the probability measures are birationally invariant.
Main theorem (B. 13)

Let $X$ be a projective variety with positive Kodaira dimension. Then the empirical measures of the canonical random point process converge in law towards the Song-Tian-Tsuji canonical measure $\mu_X$ on $X$:

$$
\frac{1}{N_k} \sum_{i=1}^{N_k} \delta_{x_i} \rightarrow \mu_X,
$$
Concretely, when $K_X > 0$ this implies the the weak convergence of the following canonical volume forms on $X$:

\[ \nu_N := \int_{X^{N-1}} \mu^{(N)} \rightarrow dV_{KE}, \]

as $N \rightarrow \infty$
Corollary

Assume that $K_X > 0$. Then the following “Bergman type” Kähler metrics

$$\omega_k := i\partial\bar{\partial} \log \int_{X^{N_k-1}} \left| (\det S^{(k)}) (\cdot, z_1 \ldots, z_{N_k-1}) \right|^{2/k} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{N_k-1} \wedge d\bar{z}_{N_k}$$

converge weakly on $X$ towards the unique Kähler-Einstein metric $\omega_{KE}$.

- More generally, when $X$ is of general type, the current $\omega_k$ is singular along the base locus of $kK_X$ and $\omega_{KE}$ is the singular Kähler-Einstein metric [Eyssidieux-Guedj-Zeriahi, ...].

- When $K_X > 0$ this is somewhat similar to the convergence of balanced Bergman metrics [Donaldson, ...]
• But one virtue here is that it also works in singular and degenerate settings.
In fact, the proof of the main theorem shows that the convergence of the empirical measure

\[ \delta_N := \frac{1}{N_k} \sum_{i=1}^{N_k} \delta_{x_i} \]

towards the deterministic measure \( \mu_X \) is exponential in the sense of large deviations.

The corresponding rate functional \( F(\mu) \) has the property that

\[ F(\omega^n/V) \]

coincides with Mabuchi’s K-energy \( M(\omega) \) of \( \omega \) (which is minimized on \( \omega_{KE} \))
The Fano case

Now consider the opposite setting when $X$ is a Fano manifold, i.e.

$$-K_X > 0$$

A KE-metric $\omega_{KE}$ must have positive Ricci curvature.

**Yau-Tian-Donaldson conjecture:** $X$ admits a KE metric iff $X$ is K-polystable

[settled by Chen-Donaldson-Sun, for $X$ smooth]
An alternative variational approach gives [B.-Boucksom-Jonsson]:

- A Fano manifold $X$ admits a unique KE metric iff $X$ is uniformly K-stable
The probabilistic approach

Recall that in the case $K_X > 0$, the probability measure on $X^{N_k}$ is defined by

$$\mu^{(N_k)} := \frac{1}{Z_{N_k}} \left| S^{(k)}(z_1, \ldots, z_{N_k}) \right|^{2/k} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{N_k} \wedge d\bar{z}_{N_k},$$

where $S^{(k)}$ is built from the $N_k$-dimensional vector space $H(X, kK_X)$.

- However, when $-K_X > 0$ the spaces $H^0(X, -kK_X)$ are trivial!

- Instead, we need to work with the spaces $H^0(X, -kK_X)$

- We are then forced to replace the power $2/k$ with $-2/k$. 


On a Fano variety $X$ it is thus natural to try to define

$$
\mu(N_k) := \frac{1}{Z_{N_k}} |S^{(k)}(z_1, \ldots, z_{N_k})|^{-2/k} \, dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{N_k} \wedge d\bar{z}_{N_k},
$$

where $S^{(k)}$ is a generator of the determinant of $H^0(X, -kK_X)$.

- However, now the normalizing constant $Z_{N_k}$ may diverge!

**Definition:** A Fano manifold $X$ is said to be *Gibbs stable* if $Z_{N_k} < \infty$, for $k \gg 1$. 
**Conjecture:** Let $X$ be a Fano manifold. Then

- $X$ is Gibbs stable iff $X$ admits a unique KE-metric $\omega_{KE}$

- In that case, $\omega_{KE}$ emerges in the many particle limit of the corresponding random point processes
Gibbs stability

The notion of Gibbs stability admits a natural algebro-geometric formulation:

Consider the following divisor $\mathcal{D}_k$ on $X^N$,

$$\mathcal{D}_k = (s^{(k)} = 0) = \{ (x_1, \ldots, x_{N_k}) : \exists s_k \in H^0(X^{N_k}, -kK_{X^{N_k}}) : s_k(x_i) = 0 \}$$

- $\mathcal{D}_k/k$ is a $\mathbb{Q}$–divisor on $X^{N_k}$ such that $\mathcal{D}_k/k \sim -K_{X^{N_k}}$

- Gibbs stability of a Fano $X$ at level $k$ means that $\mathcal{D}_k/k$ has mild singularities in the sense of the MMP.
In other words, $X$ is Gibbs stable iff

\[ \text{lct}(X^{N_k}, \frac{1}{k}D_k) > 1, \ k \gg 1 \]

This also suggests a stronger notion of Gibbs stability "uniform Gibbs stability":

\[ \liminf_{k \to \infty} \text{lct} (X^{N_k}, \frac{1}{k}D_k) > 1 \]
Using MMP techniques Fujita and Fujita-Odaka have recently shown:

\[ \text{uniform Gibbs stability} \implies \text{uniform } K\text{-stability} \]

Hence, by the resolution of the YTD

\[ \text{uniform Gibbs stability} \implies \text{existence of } \omega_{KE} \]

But the convergence of the point processes is still open (and the converse implication)
The case when $\text{Aut} \ (X)_0$ is non-trivial

If $\text{Aut} \ (X)_0$ is non-trivial, then $X$ is not Gibbs stable:

$$Z_{N_k} = \infty$$

We have to break the symmetry!

Fix a volume form $dV$ on $X$. It induces a metric $\| \cdot \|$ on $-K_X$. 
For any sufficiently small $\gamma > 0$ the ‘regularized’ probability measure on $X^{N_k}$

$$\mu_{-\gamma}^{(N)} := \frac{1}{Z_{N_k,\gamma}} \|S(k)\|^{-2\gamma/k} dV \otimes N_k$$

is well-defined when $k >> 1$.

More precisely, it is well-defined as long as $\gamma < \gamma(X)$, where the critical exponent $\gamma(X)$ is given by

$$\gamma(X) := \lim\inf_{k \to \infty} \text{lct} \left( X^{N_k}, \frac{1}{k} \mathcal{D}_k \right)$$
Conjecture

- The critical exponent $\gamma(X)$ coincides with the sup over all $\gamma$ such that Aubin’s equation
  \[
  \text{Ric} \omega = \gamma \omega + (1 - \gamma) \text{Ric} \, dV
  \]
  admits a solution $\omega_\gamma$ (assumed “minimizing” if $\gamma > 1$)

- If $\gamma < \gamma(X) \leq 1$, then the unique solution $\omega_\gamma$ emerges in the many particle limit of the corresponding random point process.

- If $X$ is “Gibbs polystable”, then one gets a Kähler-Einstein metric $\omega$ by first letting $N \to \infty$ and then increasing $\gamma \to 1$ [to be defined...]
Relations to statistical mechanics and proof strategy

Fixing a volume form $dV$ on $X$ and $\beta \in \mathbb{R}$ we consider the probability measure

$$
\mu^{(N)}_{\beta} := \frac{1}{Z_{N_k, \beta}} \| S^{(k)} \|^{2\beta/k} dV \otimes N_k \text{ on } X^{N_k}
$$

This gives the \emph{canonical} probability measures when $\beta = \pm 1$.

We can rewrite

$$
\mu^{(N)}_{\beta} := \frac{1}{Z_{N_k, \beta}} e^{-\beta E^{(N)}(x_1, \ldots, x_N)} dV \otimes N_k \text{ on } X^{N_k},
$$

where

$$
E^{(N)} := -k^{-1} \log \| S^{(k)} \|^2
$$
In general, a probability measure of the form

\[
\mu^{(N)}_{\beta} := \frac{1}{Z_{N,\beta}} e^{-\beta E^{(N)}(x_1, \ldots, x_N)} dV \otimes N_k \quad \text{on} \quad X^N,
\]

is the \textit{Gibbs measure} describing the equilibrium distribution of \( N \) particles on \( X \) with interaction energy \( E^{(N)} \) at inverse temperature \( \beta \).

- In our case the interaction \( E^{(N)}(x_1, \ldots, x_N) = -k^{-1} \log \| S^{(k)} \|^2 \) is \textit{repulsive}.

- The case when \( \beta = -\gamma < 0 \) equivalently corresponds to an \textit{attractive} interaction at inverse temperature \( \gamma \).
The starting point of the proof of the convergence when $\beta > 0$ is that

$$\frac{1}{N_k} \sum_{i=1}^{N_k} \delta_{x_i} \rightarrow \mu \implies N^{-1} E^{(N)}(x_1, \ldots, x_N) \rightarrow E(\mu),$$

in the sense of $\Gamma$–convergence [using B.-Boucksom’11]

- $E(\mu)$ is the **pluricomplex energy** of $\mu$ (i.e. $E(\mu) = (I-J)(\varphi_\mu)$)

Heuristically, this suggests the following asymptotics:
\[ \mathbb{P} \left( \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \in B_\epsilon(\mu) \right) := \frac{1}{Z_N} \int_{B_\epsilon(\mu)} e^{-\beta E^{(N)}} dV \otimes N \]

\[ \sim e^{-NF(\mu)}, \quad N \to \infty, \epsilon \to 0, \]

where the “rate functional” \( F_\beta \) is a lsc functional on \( \mathcal{P}(X) : \)

\[ F_\beta(\mu) = \beta E(\mu) + \text{Ent} \left. dV(\mu) \right| - C_\beta \]

- This is classical when \( \beta = 0 \) (i.e. \( E = 0 \)) [Boltzmann, Sanov,...]

- Thus, the probability is \underline{exponentially small}, unless \( F_\beta(\mu) = 0 \)
• $F_\beta(\omega^n) = M(\omega) := \text{Mabuchi’s K-energy.}$ So this “explains” the Chen-Tian energy+entropy formula for $M(\omega)$.

• The actual proof of the asymptotics exploits that $E^{(N)}$ is uniformly quasi-superharmonic on the orbifold $X^N/\Sigma_N$.

• To handle the case $\beta < 0$ one should exploit the plurisubharmonicity...
Supporting evidence in the Fano case ($\beta = -1$)

- The conjectures do hold when $X$ is a one dimensional log Fano.

- Then Gibbs stability is equivalent to uniform Gibbs stability and $K$-stability.

- If $X$ admits a KE metric, then the functional $F_{-1}(\mu)$ is $\text{Isc}$ on $\mathcal{P}(X)$ [BBGEZ]
The toric case and tropicalization

Let now $X$ be a toric Fano variety, i.e. $X$ is an equivariant compactification of the complex torus $\mathbb{C}^*n$.

By the usual toric dictionary

$$(X, -K_X) \longleftrightarrow P,$$

where $P$ is a certain polytope $P \subset \mathbb{R}^n$.

- Let $m_1, ..., m_{Nk}$ be an enumeration of the integer points of $kP$

- The corresponding multinomials $z^{m_i}$ span $H^0(X, -kK_X)$
Accordingly, we can write the non-normalized measure on \((\mathbb{C}^*n)^N\) as

\[
|\Delta(z_1, \ldots z_{N_k})|^{-2/k} \left( \frac{dz}{z} \wedge \frac{d\bar{z}}{z} \right) \otimes N_k
\]

where

\[
\Delta(z_1, \ldots z_{N_k}) = \sum_{\sigma \in S_N} (-1)^{\text{sign}(\sigma)} z_{1}^{m_{\sigma(1)}} \ldots z_{N_k}^{m_{\sigma(N)}}
\] (2)
Exploiting the torus symmetry

Recall that, modulo Aut (X)₀ any Kähler-Einstein metric ω_{KE} on a toric variety X is invariant under the action of the real torus T.

In particular, its volume form dV_{KE} on C*ⁿ "descends" to Rⁿ under

\[
\text{Log : } C^*^n \to R^n, \ z \mapsto x := (\log |z_1|^2, \ldots, \log |z_n|^2),
\]

\[
dV_{KE} = e^{-\psi(z)} \frac{dz}{z} = e^{-\phi(x)} \, dx \wedge dy,
\]

where dy is the invariant top form on the torus fiber T

\[
\psi(z) \text{ psh } \longleftrightarrow \phi(x) \text{ convex}
\]
However, the measure

\[ |\Delta(z_1, \ldots z_{N_k})|^2/k \left( \frac{dz}{z} \wedge \frac{d\bar{z}}{\bar{z}} \right)^\otimes N_k \]

does not “descend” to \((\mathbb{R}^n)^N\), i.e.

\[ E(z_1, \ldots z_{N_k}) := \log |\Delta(z_1, \ldots z_{N_k})|^2 \]

is not \(T^N\)-invariant!
Tropicalization

*The philosophy of Tropicalization:* replace an elusive problem for polynomials in \( \mathbb{C}^d \) with a simpler one, for piece-wise affine convex functions in \( \mathbb{R}^d \):

\[
\sum_{m \in P} c_m z^m \leadsto \phi(x) := \max_{m \in P} x \cdot m
\]

Equivalently, the psh function on \( \mathbb{C}^d \)

\[
\Psi(z) := \log |P(z)|
\]

is replaced by a convex function on \( \mathbb{R}^d \):

\[
\phi(x) := \lim_{\lambda \to \infty} \lambda^{-1} \Psi(e^{\lambda x}, e^{iy})
\]
Applying this philosophy here suggests studying the point processes on $\mathbb{R}^n$ defined by the following measure on $(\mathbb{R}^n)^N$:

$$e^{-E_{trop}(x_1,\ldots,x_N)}dx\otimes N,$$

where

$$E_{trop}(x_1,\ldots,x_N) := \max_{\sigma \in S_N} \left( x_1 \cdot m_{\sigma(1)} + \ldots + x_N \cdot m_{\sigma(N)} \right),$$

$m_1,\ldots,m_N$ are the lattice points in $kP$.

- **Optimal transport interpretation:** $-E_{trop}(x_1,\ldots,x_N)$ is the minimal quadratic cost of transporting the measure $\delta_N(x)$ to $\delta_N(m)$.
The action of $T_C \subseteq \text{Aut} \ (X)_0$ on $X$ corresponds to the translation action on $\mathbb{R}^n$.

It forces

$$Z_N := \int e^{-E_{trop}(x_1, \ldots, x_N)} \, dx \otimes N = \infty$$

for any polytope $P$.

Again, we have to break the symmetry: $E_{trop}(x_1, \ldots, x_N) \rightsquigarrow$

$$\gamma E_{trop}(x_1, \ldots, x_N) + (1 - \gamma)(\phi_P(x_1) + \ldots + \phi_P(x_N))$$

Then $Z_{N, \gamma}$ is finite for $\gamma < \gamma_P$.
**Thm** (B.- Önnheim):

- The critical exponent \( \gamma_P \) coincides with the invariant \( R(X_P) \)

- When \( \gamma < \gamma_P \) the unique solution \( \omega_\gamma \) of the Aubin-Yau equation emerges in the many particle limit of the corresponding random point process.

- When \( X_P \) is K-polystable one gets a KE metric \( \omega_{KE} \) by first letting \( N \to \infty \) and then \( \gamma \to 1 \).

The proof exploits convexty of \( E_{trop} \), i.e. that the corresponding probability measures are log-concave (Prekopa inequality, Borell’s inequality, ...).
Thank you!!!